PHY 456: "Lecture 2"
Quantum Mechanical Scattering.

Recall: probability current:

$$\vec{j}(\vec{r},t) = -i\frac{\hbar}{2\mu}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right).$$

If we let $\Psi(\vec{r},t) = |\Psi| e^{i\arg\Psi}$, then

$$\vec{j}(\vec{r},t) = \frac{\hbar}{\mu}|\Psi|^2\,\nabla(\arg\Psi).$$ (see HW 1)

We have our scattering state $\Psi \to \varphi^{SCATT}$:

$$\boxed{\varphi^{SCATT}\Big|_{r\to\infty} = e^{ikz} + f_k(\theta,\varphi)\frac{e^{ikr}}{r}}$$

amplitude is dependent on k.

→ If we imagine an experiment where many particles are prepared in the state $|\varphi^{SCATT}\rangle$ (identical velocities) (or we repeat experiment many times) then

incoming flux I = # of particles crossing area $\perp \hat{z}$ in unit time.

so $I = A \, j_z [e^{ikz}]$ ← At $r \to \infty$ limit, e^{ikz} dominates

↑ some constant

In other words, for the incoming beam $\varphi^{SCATT} = e^{ikz}$,

$j_z \sim$ "amount of probability" crossing unit area $d\vec{A} \perp \hat{z}$ in dt

↖ '=' implies that we need to include normalization.

and clearly, all being probabilistic, particle # flux ~ probability flux.

$$\implies I = A \cdot \frac{\hbar}{\mu} k .$$

Choose A to have $[I] = \frac{\text{\# particles}}{\text{area} \cdot \text{time}}$, cancels in σ!

- What about flux in $d\Omega = \sin\theta \, d\theta \, d\varphi$?

Here, we must take into account $f_k(\theta, \varphi) \frac{e^{ikr}}{r}$ only.

PHYSICALLY: beam has finite width. We look at $(\theta, \varphi)_{r \to \infty}$ such that $d\Omega$ is outside beam.

MATH: $r \to \infty$ interference term vanishes between e^{ikz} & $f_k(\theta, \varphi) \frac{e^{ikr}}{r}$ vanishes when integrated over small $d\Omega$ apart from when $\theta = 0$ (see HW).

(Highly oscillating, due to Riemann-Lebesgue Lemma)

It makes sense that as $r \to \infty$ the component of

$\psi = f_k(\theta,\varphi)\,\frac{e^{ikr}}{r}$ is in the $\hat{r}$ direction (other components die off faster than $\frac{1}{r}$ - do it yourself) so

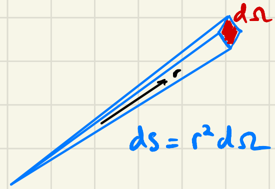
$$j_r = \frac{\hbar}{\mu}\,|\psi|^2\;\vec{r}\cdot\nabla(\arg\psi)$$

since $\psi \sim e^{i\arg f_k(\theta,\varphi) + ikr}$

$$= \frac{\hbar}{\mu}\,\frac{|f_k(\theta,\varphi)|^2}{r^2}\,\frac{\partial}{\partial r}\left(kr + \cancel{\arg f_k(\theta,\varphi)}\right)$$

vanishes $\frac{\partial}{\partial r} f_k(\theta,\varphi) = 0$.

$$= \frac{\hbar}{\mu}\,\frac{k}{r^2}\,|f_k(\theta,\varphi)|^2.$$



So probability flux through $ds = r^2 d\Omega$ is

similar to $\sim |j_r|\,ds = \frac{\hbar}{\mu}\,k\,|f_k(\theta,\varphi)|^2\,d\Omega$

normalization for '='

and $dN(\theta,\varphi)$ = # particles crossing $d\Omega|_{r=\infty}$ in unit time.

$$= A\,\frac{\hbar}{\mu}\,k\,|f_k(\theta,\varphi)|^2\,d\Omega. \quad \text{(i)}$$

Same constant as before in I units.

Our definition of $d\sigma(\theta,\varphi)$ was $= \dfrac{\#\text{ particles through } d\Omega}{\text{incoming flux}}$

with $dN(\theta,\varphi) = I\,d\sigma(\theta,\varphi)$, (ii)

hence

$$I = A\cdot\frac{\hbar}{\mu}k. \quad \text{(iii)}$$

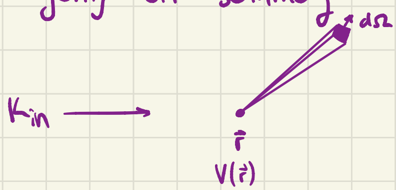
Using (i), (ii), (iii) we find

$$\boxed{\begin{aligned} d\sigma(\theta,\varphi) &= |f_K(\theta,\varphi)|^2\, d\Omega \\ \text{(or)}\quad \frac{d\sigma}{d\Omega} &= |f_K(\theta,\varphi)|^2 . \end{aligned}}$$

Most important result thus far!

Before going on: summary

[1]



[2]

$$\varphi^{SCATT}(r)\Big|_{r=\infty} = e^{ikz} + \frac{f_K(\theta,\varphi)}{r} e^{ikr}$$

Stationary scattering solution, physically motivated.

[3]

$f_K(\theta,\varphi)$ determines $\dfrac{d\sigma(\theta,\varphi)}{d\Omega} = |f_K(\theta,\varphi)|^2$

$=$ # events in $d\Omega$ / unit time.

$\equiv$ "Differential Scattering Cross Section"

- BORN SERIES APPROXIMATION

→ Determine $\varphi(\vec{r})$ from scattering equation $(\nabla^2 + k^2 - \mathcal{U}(\vec{r}))\,\varphi(\vec{r}) = 0$

given by $\varphi(\vec{r})\Big|_{r\to\infty} = e^{ikz} + \dfrac{f_K(\theta,\varphi)}{r} e^{ikr}$.

→ Use approximation methods (perturbation theory)

We have that

$$\left(\nabla^2 + k^2 - \mathcal{U}(\vec{r})\right)\varphi(\vec{r}) = 0$$

$$\Longrightarrow \underbrace{\left(\nabla^2 + k^2\right)\varphi(\vec{r}) = \mathcal{U}(\vec{r})\,\varphi(\vec{r})}.$$

TRICK: turn this into an integral equation

and find iterative solution where finite

terms are taken $\equiv$ "BORN SERIES".

Use Green's function:

$$\left(\nabla^2 + k^2\right) G(\vec{r}) = \delta^{(3)}(\vec{r}).$$

CLAIM: $G_\pm(\vec{r}) = -\frac{1}{4\pi r} e^{\pm ikr}$ $\left(\pm \equiv \begin{matrix}\text{incoming} \\ \text{outgoing}\end{matrix}\,?\right)$

$$\nabla^2 G_+ = -\left(\nabla^2 \tfrac{1}{r}\right)e^{ikr}\cdot\frac{1}{4\pi} - \frac{1}{4\pi r}\left(\nabla^2 e^{ikr}\right) - 2\left(\nabla \tfrac{1}{4\pi r}\right)\left(\nabla e^{ikr}\right)$$

$\downarrow -\nabla^2\left(\frac{1}{4\pi r}\right) = \delta(\vec{r})$

$$= \delta(\vec{r})\, e^{ikr} \quad - \frac{1}{4\pi r}\left(-k^2 + \frac{2ik}{r}\right)e^{ikr} \quad + \frac{2ik}{4\pi r^2} e^{ikr}$$

Thus $\left(\nabla^2 + k^2\right)\left(-\frac{e^{ikr}}{4\pi r}\right) = \delta(\vec{r})$

$$= \delta(\vec{r})\,e^{ikr} + \frac{k^2}{4\pi r}e^{ikr} - \frac{2ik}{4\pi r^2}e^{ikr} + \frac{2ik}{4\pi r^2}e^{ikr} - \frac{k^2}{4\pi r}e^{ikr}$$

$$= \delta(\vec{r})\,e^{ikr}$$

$$= \delta(\vec{r}) \;\checkmark$$

with the solution $G_\pm(\vec{r}) = -\frac{1}{4\pi r} e^{\pm ikr}$.

How does this help? Let

$$(\nabla^2 + k^2)\,\varphi(\vec r) = H(\vec r)$$ ← some function H (for inhomogeneity)

$$\Downarrow$$

$$\varphi(\vec r) = \varphi_0(\vec r) + \int d^3r'\, H(\vec r')\, G(\vec r - \vec r')$$

↑ Solution to homogeneous equation.
$(\nabla^2 + k^2)\,\varphi_0(\vec r) = 0$
when $\mathcal{U}(\vec r) = 0$.

$$\Longrightarrow (\nabla^2 + k^2)\int d^3r'\, H(\vec r')\, G(\vec r - \vec r') = (\vec r)$$

$$= \int d^3r'\, H(\vec r')\, \underbrace{(\nabla_r^2 + k^2)\, G(\vec r - \vec r')}_{\delta(\vec r - \vec r')} = (\vec r)$$

$$= \int d^3r'\, H(\vec r')\, \delta^{(3)}(\vec r - \vec r') = (\vec r) \quad \checkmark.$$

Thus the complete solution for $\varphi(\vec r)$ is (for $H(\vec r) = \mathcal{U}(\vec r)\,\varphi(\vec r)$)

$$\varphi(\vec r) = \varphi_0(\vec r) + \int d^3r'\, G_+(\vec r - \vec r')\,\varphi(\vec r')\,\mathcal{U}(\vec r')$$

↑ G_+ because we want outgoing waves post-scattering.

Outgoing $= e^{ikr}$ plane wave.
But missing e^{ikz} for incoming on $\varphi^{SCATT}(\vec r)$.
→ Can fix by demanding it be there

$$\boxed{\varphi^{SCATT}(\vec r) = e^{ikz} + \int d^3r'\, G_+(\vec r - \vec r')\,\varphi^{SCATT}(\vec r')\,\mathcal{U}(\vec r')}$$

SCATTERING SOLUTION
(LEADING ORDER IN BORN APPROX.)

Our goal is to iterate the solution to determine a perturbative form:

Perturbate: Imagine $\mathcal{U}(\vec{r})$ is "small":

$$\varphi^{SCATT}(\vec{r}) = e^{ikz} + \text{'small'} \ldots$$

$$\varphi^{SCATT}(\vec{r}) = e^{ikz} + \int d^3r' \, G_+(\vec{r}-\vec{r}') \, \mathcal{U}(\vec{r}') \, e^{i\vec{k}\cdot\vec{r}'}$$ LEADING ORDER

$$+ \int d^3r' \, d^3r'' \, G_+(\vec{r}-\vec{r}') \mathcal{U}(\vec{r}') G_+(\vec{r}'-\vec{r}'') \mathcal{U}(\vec{r}'') e^{i\vec{k}\cdot\vec{r}''}$$

$$+ \int d^3r' d^3r'' d^3r''' \, G_+(\vec{r}-\vec{r}') \, \mathcal{U}(\vec{r}') \, G_+(\vec{r}'-\vec{r}'') \, \mathcal{U}(\vec{r}'') \, G_+(\vec{r}''-\vec{r}''') \, \mathcal{U}(\vec{r}''') \, e^{i\vec{k}\cdot\vec{r}'''}$$

$$+ \ldots$$

But we aim to only use leading order term in PHY456.

→ Leading order term ( ) gives good approximation and a closed-form solution (we'll see - very useful)

Thus:

(LEADING ORDER)

$\vec{k}\cdot\vec{r}' \equiv kz'$ ($\vec{k} = k\hat{z}$ only)

$$\varphi^{SCATT}(\vec{r}) = e^{ikz} + \int d^3r' \, G_+(\vec{r}-\vec{r}') \, e^{ikz'} \, \mathcal{U}(\vec{r}')$$

$$= e^{ikz} - \frac{1}{4\pi} \int d^3r' \, \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \, e^{ikz'} \, \mathcal{U}(\vec{r}') .$$

Still complex, but recall that we're after the solution

$$\varphi^{SCATT}(\vec{r})\Big|_{r\to\infty} = e^{ikz} + f_k(\theta,\varphi) \frac{e^{ikr}}{r} .$$

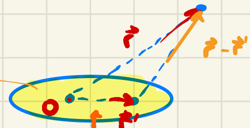
Take $r \to \infty$ in integrand:

* $\dfrac{1}{|\vec{r}-\vec{r}'|} \longrightarrow \dfrac{1}{r}$

+ Exponent $e^{ik|\vec{r}-\vec{r}'|}$ we have $|\vec{r}-\vec{r}'| \equiv$ projection of $\vec{r}$ on $\vec{r}'$

So at $\vec{r} >> u(\vec{r}')$:

outside of bounds of u



$$|\vec{r}-\vec{r}'| \simeq r - \frac{\vec{r}\cdot\vec{r}'}{r}$$

$$= r - \hat{r}\cdot\vec{r}'$$

Region where $u(\vec{r}')$ is 'Substantial': has source of finite size, so $\vec{r}'$ is limited to this region.

<u>THUS</u>

$$\varphi^{SCATT}(\vec{r}) = e^{ikz} - \frac{e^{ikr}}{4\pi r}\int d^3r'\, e^{i(kz' - \hat{r}\cdot\vec{r}'k)}\, u(\vec{r}')$$

A CLOSED FORM: ONLY DEPENDS ON $\hat{r}(\theta,\varphi)$!

which, when compared to $\varphi^{SCATT}(\vec{r}) = e^{ikz} + f_k(\theta,\varphi)\,\frac{e^{ikr}}{r}$

we find

$$\boxed{f_k(\theta,\varphi) = -\frac{1}{4\pi}\int d^3r'\, e^{i(\overset{(in)}{k} z' - \overset{(out)}{k}\hat{r}(\theta,\varphi)\cdot\vec{r}')}\, u(\vec{r}')}$$

BORN APPROXIMATION FOR SCATTERING AMPLITUDE $f_k(\theta,\varphi)$.

Note that this is written with two different wave vectors $\vec{k}^{(in)} = k\hat{z}$ and $\vec{k}^{(out)} = k\hat{r}(\theta,\varphi)$ since a momentum change has a high probability of occuring.

It follows that, since $u(\vec{r}) \equiv \frac{2\mu}{\hbar^2}V(\vec{r})$,

$$\boxed{\frac{d\sigma_{BORN}}{d\Omega} = \frac{\mu^2}{4\pi^2\hbar^4}\left|\int d^3r\, V(\vec{r})\, e^{i\vec{r}\cdot(\vec{k}^{(in)} - \vec{k}^{(out)}(\theta,\varphi))}\right|^2}$$

which can be interpreted as the Fourier Transform of $V(\vec{r})$ with respect to momentum transfer.

Important question when invoking perturbation theory:

Obviously, this may be evaluated for any $V(\vec{r})$ where $\int d^3\vec{r}$ converges... but when is this to be believed?

(Will eventually be addressed)...

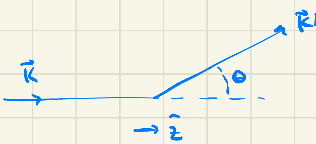
- Example: Yukawa Potential $V(r) = \alpha \frac{e^{-r/\ell}}{r}$ (ℓ = screening length)

QFT $V(r) = -\frac{g^2}{4\pi r} e^{-M_\phi r}$

g: coupling interaction constant.

→ Describes mesons, "Screened Coulomb"...

Since V is spherically symmetric, we expect no φ-dependence in $\hat{r}(\theta,\varphi)$ or $d\sigma(\theta,\varphi)$.



θ: scattering angle, do not confuse w/ integration variable $\vec{r}'$.

Better to write: $$\int d^3r'\, V(\vec{r}')\, e^{-ikz' + ik\hat{r}(\theta,\varphi)\cdot\vec{r}'}$$

$$\equiv \int d^3r'\, V(\vec{r}')\, e^{-i\vec{K}'\cdot\vec{r}'}$$ $\vec{K}' \equiv \vec{k}^{(in)} - \vec{k}^{(out)}(\theta,\varphi)$

Since $V(\vec{r})$ is direction-independent, we can choose a convenient axis such that $\vec{K}' \parallel \hat{z}'$:

$$\int d^3r'\, V(\vec{r}')\, e^{-ik'r'\cos\theta'} = \int_0^\infty \int_{-1}^{1} \int_0^{2\pi} dr'\, d\cos\theta'\, d\varphi \cdot \alpha \frac{e^{-r'/\ell}}{r'}\, e^{-ik'r'\cos\theta'}$$

$$= \int_0^\infty dr'\; 2\pi \cdot \left(\frac{1}{-ik'r'}\, e^{-ik'r'\cos\theta'} \Big|_{\cos\theta'=-1}^{\cos\theta'=1} \right) \cdot \left(r'^2 \alpha \frac{e^{-r'/\ell}}{r'} \right)$$

$$= 2\pi \cdot \frac{1}{-ik'}\, \alpha \int_0^\infty dr'\, e^{-r'/\ell} \left(e^{-ik'r'} - e^{+ik'r'} \right)$$

$$= -\frac{2\pi\alpha}{ik'} \int_0^\infty dr' \left[e^{-r'/\ell - ik'r'} - e^{-r'/\ell + ik'r'} \right]$$

Let $r' = \ell t$: $dr' = \ell\, dt'$, $dt' = \frac{dr'}{\ell}$

$$= -\frac{2\pi\alpha\ell}{ik'} \int_0^\infty dt \left[e^{-t(1+ik'\ell)} - e^{-t(1-ik'\ell)} \right]$$

$\int_0^\infty ds\, e^{-as} = \frac{1}{a}$! $\longrightarrow$

$$= \frac{2\pi i\alpha\ell}{k'} \left[\frac{1}{1+ik'\ell} - \frac{1}{1-ik'\ell} \right]$$

$$= \ldots$$

$$= 4\pi\alpha\ell^2 \cdot \frac{1}{1+k'^2\ell^2}$$

$k' \equiv |\vec{k}^{(in)} - \vec{k}^{(out)}|$

So $k'^2 \equiv k^{(in)2} + k^{(out)2} - 2\vec{k}^{(in)} \cdot \vec{k}^{(out)}$

$= 2k^2(1-\cos\theta)$

Since momentum must be conserved, and scattering angle is θ.

$$= 4\pi\alpha\ell^2 \cdot \frac{1}{1+2\ell^2k^2(1-\cos\theta)} \cdot$$

$1-\cos\theta = 2\sin^2\frac{\theta}{2}$

Back to Born approximation:

$$d\sigma_B = |f_k(\theta,\varphi)|^2 d\Omega$$

$$= \frac{\mu^2}{4\pi^2\hbar^4}\left|\frac{4\pi\alpha\ell^2}{1+4\ell^2k'^2\sin^2(\frac{\theta}{2})}\right|^2 2\pi\sin\theta\, d\theta .$$

Again, for $V = \alpha\frac{e^{-r/\ell}}{r}$; as $\ell \to \infty$: coulomb

and $\alpha = zz'e^2$ (Bohr)

$$d\sigma_B(\theta) = \frac{\mu^2}{4\pi^2\hbar^4}\frac{16\pi^2(zz'e^2)^2}{16k^4\sin^4\frac{\theta}{2}}$$

$$= \frac{\mu^2}{4\hbar^2}\frac{(zz'e^2)^2}{k^4\sin^4\frac{\theta}{2}}$$

$E = \frac{\hbar^2k^2}{2\mu}$; $k^4 = \frac{4\mu^2E^2}{\hbar^4}$

$$= \frac{(zz'e^2)^2}{16E^2\sin^4(\frac{\theta}{2})}$$

Rutherford's Formula!

The Yukawa result:

$$d\sigma_B = \frac{\mu^2}{4\pi^2\hbar^4}\left|\frac{4\pi\alpha\ell^2}{1+4\ell^2k'^2\sin^2(\frac{\theta}{2})}\right|^2 2\pi\sin\theta\, d\theta .$$

- At $k \ll \frac{1}{\ell}$ range, it becomes isotropic

(de Broglie wavelength $E = \frac{\hbar^2k^2}{2\mu}$, $p = \hbar k$

recall $\lambda_{db} \sim \frac{\hbar}{p}$)

$\Rightarrow \lambda \gg \ell$ is much greater than the range of potential

• On the other hand, angular dependence is shortest for $\lambda << \ell$ and "beamed" near $\theta \sim 0$.

But when is the Born Approximation "Good"?

↳ Is good if 'small' term is << 'leading' term.

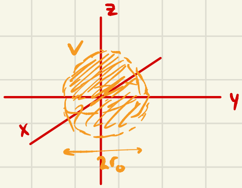
i.e. $$\frac{f_k(\theta,\varphi)\, e^{ikr}}{r} << e^{ikz} \qquad \{ r < r_0 \quad (r_0 \equiv \text{range of } V) \}$$

$$\left| \int dr'\, G_+(r-r')\, u(r')\, e^{ikz'} \right| << \left| e^{ikz} \right| \qquad \{ r \simeq r_0 \}$$

$$\Longrightarrow \left| \frac{1}{4\pi} \int d^3r'\, \frac{e^{ikr'}}{r'}\, u(r')\, e^{i\vec{k}^{(in)}\cdot\vec{r}'} \right| << 1$$

IF $u(\vec{r}') = u(r') = \frac{2\mu}{\hbar} V(r')$ (spherical)

$$\Longrightarrow \frac{2\mu}{\hbar^2 k} \left| \int dr'\, e^{ikr'} \sin kr'\, V(r') \right| << 1$$



$2r_0$: depth/width/height of V

= size where $V(r)$ is "appreciable".

Limit Analysis:

i) Low energy limit $\left(k \to 0 \; ; \; V_0 << \frac{\hbar^2}{\mu r_0^2} \right)$

• V_0 << energy of particle of deBroglie wavelength r_0

(if attractive, V shouldn't bind other particles in the r_0 region)

2) High energy limit $\left(V_0 \ll \frac{\hbar^2}{\mu r_0^2} (k r_0) \right)$ judiciously drop oscillation term

- If good at low k, also good at high k.

Next: angular momentum for central potentials.